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Semidirect products of categories and applications [1]

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Abstract

This paper offers definitions for the semidirect product of categories, Cayley graphs for categories, and the kernel of a relational morphism of categories which will allow us to construct free (profinite) objects for the semidirect product of two (pseudo) varieties of categories analogous to the monoid case. The main point of this paper is to prove $g(\mathbf{V} * \mathbf{W}) = g\mathbf{V} * g\mathbf{W}$. Previous attempts at this have contained errors which, the author feels, are due to an incorrect usage of the wreath product. In this paper, we make no use of wreath products, but use instead representations of the free objects. Analogous results hold for semigroups and semigroupoids.

We then give some applications by computing pseudoidentities for various semidirect products of pseudovarieties. A further application will appear in a forthcoming joint work with Almeida where we compute finite iterated semidirect products such as $\mathbf{A} * \mathbf{G} * \mathbf{J} * \mathbf{G} * \mathbf{J} * \cdots * \mathbf{G} * \mathbf{J}$. We obtain analogous results for the two-sided semidirect product. This paper also recovers the incorrectly proven result of Jones and Pustejovsky, needed to prove that $\mathbf{DS}$ is local.

We conclude by showing that our semidirect product can be used to classify split extensions of groupoids, generalizing the classical theory for groups. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

There have been various proposals for the semidirect product of categories and pseudovarieties of categories. The main goal has always been to show $g\mathbf{V} * g\mathbf{W} = g(\mathbf{V} * \mathbf{W})$ and the two-sided analog. However both [11, 18], have acknowledged flaws in their arguments. [2] Essentially, this is due to various incorrect notions of the definition of

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[2] Errors in both proofs have been pointed out by L. Teixeira.

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a wreath product of categories. We will propose a reasonably natural definition of a semidirect product on the varietal level and show that our definition is equivalent to that of Jones and Pustejovsky [11]. To do this, we will construct free objects (and free profinite objects in the pseudovariety case). We will then use these free objects to obtain the above equality. All the results will apply to the case of semigroupoids as well (via the same types of arguments needed to go from monoids to semigroups in [20]). We will then give some applications to computing the global of various semidirect products of monoids. As a final application of our semidirect product, we classify the split extensions of a groupoid.

To be fair, many of the ideas are contained in Tilson's [21], and Almeida and Weil's [6], but one must come up with the correct generalizations of the definitions for the old machinery to work. The key new ideas are that we only truly have a derived category theorem (relating semidirect products with derived categories) on the variety level and that one should define the semidirect product of categories so that the free objects have the accustomed form. All our results will clearly hold for the reverse-semidirect product. These results naturally extend to the two-sided semidirect product, but for simplicity of notation, we handle only the one-sided case in detail, remarking on the changes necessary for the two-sided case in Section 9.

For undefined notation on categories, see [20, 6], and in general, [1]. We assume as in [1], that all semidirect products are unitary. In this paper, we do not allow empty algebraic objects. In this paper, we will generally write morphisms on the right of their arguments.

2. A semidirect product

We begin with a proposal for the semidirect product of two categories. Note, we will mention explicitly semigroupoids only when the arguments are different. All theorems throughout apply equally well for semigroupoids.

First recall, a *directed graph* X consists of a pair of sets $V(X)$ of vertices, $E(X)$ of edges, and maps $\alpha, \omega : E(X) \to V(X)$ which select the initial and terminal vertices, respectively. We will write $X(x,x')$ for the set of edges from x to x'. One can then view a set as a one vertex graph. A *semigroupoid* is a graph with an associative multiplication defined on composable edges. A *category* is a semigroupoid which has local identities at each vertex, see [20, 6] for more on categories viewed as algebraic objects. Edges of categories and semigroupoids will also be called arrows, while vertices will sometimes be called objects. The collection of edges between two vertices c_1, c_2 of a category C is called a *hom set* and is written $C(c_1, c_2)$. We often will view monoids as one object categories and semigroups as one object semigroupoids. In this paper, we will call maps between categories *morphisms* instead of functors to emphasize the analogy between categories and monoids. A morphism is called *faithful* if it is injective when restricted to hom sets, *full* if it is surjective when restricted to hom sets, and a *quotient* when it is full and bijective on object sets. Quotient morphisms correspond to taking a quotient category by a congruence, see [15, 20]. These terms are also used for graph morphisms.

To motivate the definition of the action of a category C on a category D, we first define the Cayley graph of a category with respect to a generating graph X. Recall, a category C is generated by a graph X if there is a graph morphism $\varphi : X \to C$, such that the smallest subcategory containing $X\varphi$ is C. One often blurs the distinction between x and $x\varphi$. Then we define the *Cayley graph* of C with respect to X, $\Gamma_X(C)$, by

$$V(\Gamma_X(C)) = E(C),$$

where in the semigroupoid case one adds the local identities, and

$$E(\Gamma_X(C)) = E(C) \times_{\omega,\alpha} E(X) = \{(e,x) \in E(C) \times E(X) \mid e\omega = x\alpha\}.$$

The initial vertex of an edge (e,x) is e and the terminal vertex is ex. In pictures the arrows look like

$$e \xrightarrow{x} ex.$$

For the case of monoids there is a natural left action of C on $\Gamma_X(C)$, induced by left multiplication on the vertices. But in our case, there is only a partial action. Namely if $f : c_1 \to c_2$, then $f\cdot$ is defined on all vertices $g : c_2 \to c_3$. Furthermore, this action has the properties:

1. If f, h are composable, then $(fh)g = f(hg)$ whenever either side makes sense;
2. If f, h are composable, then hg defined implies $(fh)g$ is defined;
3. Identities act as partial identities.

Due to the well-known connection between free objects for semidirect products, derived categories, and Cayley graphs, the following definition should be more palatable in lieu of the above.

Let C be a category. We define $\mathrm{PFEnd}_{\mathrm{L}}(C)$ to be the monoid of *partial full endofunctors* of C, acting on the left, under the standard composition of partial maps. That is the monoid of functors, acting on the left, whose domains are full subcategories and codomains are arbitrary subcategories of C. Note we do not allow empty domains. A *left action* of a category D on a category C is a map $\rho : E(D) \to \mathrm{PFEnd}_{\mathrm{L}}(C)$ satisfying the following properties:

- If fg is defined, $(fg)\rho = f\rho g\rho$ (ρ is a partial homomorphism).
- If fg is defined $dom(g\rho) \subseteq dom((fg)\rho)$ where *dom* means domain ($\mathscr{L}$-consistency of domain). Note that in conjunction with the above requirement, we see that $dom(g\rho) = dom((fg)\rho)$.
- $1_c\rho$ must be a partial identity (in the semigroupoid case this requirement says the action is unitary and we will only consider such actions).

For topological categories, we require the induced partial maps $E(D) \times V(C) \to V(C)$ and $E(D) \times E(C) \to E(C)$ to be continuous when restricted to the domain of definition. One can define right actions and two-sided actions analogously.

To form a semidirect product of categories C and D (we will write C additively for convenience, but do not assume commutativity of any sort) we need a left action ρ

of D on C, often we will write $f\cdot$ for $f\rho$, and a (continuous) map $\psi : V(D) \to V(C)$, called the basepoint map, such that for $g \in E(D)$, $g : d_1 \to d_2$, $g(d_2\psi)$ is defined. The *semidirect product* $C *_{\psi,\rho} D$, written $C * D$ when the action and basepoint maps are understood, is defined by

- $V(C * D) = V(D)$.
- $E(C * D) = \{(f, g) \in E(C) \times E(D) \mid g : d_1 \to d_2,\ f : d_1\psi \to g(d_2\psi)\}$.
- The initial and terminal vertices for (f, g) are as for g.
- The identity for $d \in V(D)$ is $(0_{d\psi}, 1_d)$.
- Given $g : d_1 \to d_2$, $g' : d_2 \to d_3$, $f : d_1\psi \to g(d_2\psi)$, and $f' : d_2\psi \to g'(d_3\psi)$, we define $(f, g)(f', g') = (f + gf', gg')$.

It is easy to check that the definition of a left action and the requirement on ψ guarantees that this is a well defined, associative multiplication and that the proposed identities are indeed identities. Furthermore if C and D are topological categories, it is easy to verify that $C * D$ is as well.

We remark this definition includes the standard semidirect product of monoids. The semidirect product of a category with a monoid considered in [6] is a special case of this construction where the left action is always fully defined. Note that there is a natural (continuous) projection morphism to D. However, although the projection is bijective on vertices, it need not be full. This is already true in the semidirect product defined in [6]. However, the free object will turn out to have a quotient projection map.

3. Basic properties of semidirect products

The following are analogs of well-known results on semidirect products of monoids.

Proposition 3.1. *Let $C_i *_{\rho_i,\psi_i} D_i$ be semidirect products. Then $\prod D_i$ has a natural action on $\prod C_i$, via $\prod \rho_i$, and, $\prod (C_i *_{\rho_i,\psi_i} D_i) \cong \prod C_i *_{\prod\rho_i,\prod\psi_i} \prod D_i$.*

Proof. The argument is identical to the one for monoids and so we omit it. $\square$

We also have the following functorial properties.

Proposition 3.2. *Let $\varphi : C \to C'$, $\varphi' : D \to D'$ be morphisms and $C *_{\rho,\psi} D, C' *_{\rho',\psi'} D'$ be semidirect products such that*

1. $\psi\varphi = \varphi'\psi'$.
2. *For an object c of C and an arrow g of D, if gc is defined, $g\varphi'(c\varphi)$ is defined and $(gc)\varphi = g\varphi'(c\varphi)$.*
3. *For an arrow f of C and an arrow g of D, if gf is defined, then by the above condition $g\varphi' f\varphi$ is defined and, we require $(gf)\varphi = g\varphi' f\varphi$.*

In this case we say φ and φ' are compatible with the semidirect product actions. Define $\tilde{\varphi} : C * D \to C' * D'$ by φ' on objects and $\varphi \times \varphi'$ on arrows. Then $\tilde{\varphi}$ is a *morphism. Furthermore, if φ, φ' are both faithful or both quotient, $\tilde{\varphi}$ is also faithful or quotient respectively.*

Proof. Let $g: d_1 \to d_2$, $f: d_1\psi \to gd_2\psi$. Then an easy verification shows

$$g\varphi' : d_1\varphi' \to d_2\varphi'$$

and

$$f\varphi : d_1\psi\varphi = d_1\varphi'\psi' \to (gd_2\psi)\varphi = g\varphi' d_2\varphi'\psi'.$$

So $\tilde{\varphi}$ is a graph morphism. The remainder of the verification proceeds exactly as in the well-known monoid case. The last statement is trivial from the definition of $\tilde{\varphi}$. □

For a semidirect product $C * D$, in the spirit of Tilson [21], we define the *core hom sets* of C to be those hom sets which are actually used for the semidirect product, that is of the form $C(d_1\psi, gd_2\psi)$ where ψ is the basepoint map and $g: d_1 \to d_2$. The following result is of a similar flavor to the previous one. First recall that a relational morphism of categories C and D is a subcategory of $C \times D$ such that the restriction of the projection to C is a quotient morphism, see the beginning of Section 4 or [20] for more.

Proposition 3.3. *Let $\varphi : C \to C'$ be a relational morphism and $C *_{\rho,\psi} D$, $C' *_{\rho',\psi'} D$ be semidirect products such that*

1. $\psi\varphi = \psi'$.
2. *For an object c of C and arrow g of D, if gc is defined, $g(c\varphi)$ is defined and $(gc)\varphi = g(c\varphi)$.*
3. *For an arrow f of C and arrow g of D, if gf is defined, then by the above condition g is defined on $f\varphi$ and we require $g(f\varphi) \subseteq (gf)\varphi$.*

In this case we say φ is (D)-compatible with the semidirect product actions. If φ is a morphism, this is the same as saying φ and id_D are compatible with the semidirect product actions in the above sense. Define $\tilde{\varphi} : C*D \to C'*D$ by the identity on objects and $\varphi \times id_{E(D)}$ on arrows. Then $\tilde{\varphi}$ is a relational morphism. We say φ is injective on the core if when restricted to any core hom set of C, φ is an injective relation. In this case $\tilde{\varphi}$ is a division.

Proof. If φ is injective on core hom sets, it is trivial to see $\tilde{\varphi}$ is injective. We see that the image of an arrow is in the correct hom set by the same argument as above. The rest of the argument proceeds as in the monoid case [16]. □

We leave it to the reader to verify that the class of (D)-compatible morphisms is closed under composition and inverting quotient morphisms.

The following generalizes the fact that the trivial monoid acts as an identity for $*$ on the monoid level.

Proposition 3.4. *Let C be a category and let K_C be the category which is the complete graph on $V(C)$, that is has the same vertex set as C and exactly one edge between any two (not necessarily distinct) vertices, with the unique composition function. Then there are actions so that $K_C * C \cong C$ and $C * K_C \cong C$.*

Proof. First let C act trivially on K_C and let the basepoint map be the identity. It is easy to see the projection $\pi : K_C * C \to C$ is an isomorphism. Now let K_C act trivially on C with the basepoint map the identity. Then $V(C * K_C) = V(C)$. Also $(C * K_C)(c_1, c_2) = \{(f,g) \mid g : c_1 \to c_2 \in K_C, f : c_1 \to c_2 \in C\}$. Since K_C acts trivially on C and has a unique arrow between any two vertices, it is easy to see that the graph morphism which is identity on vertices and projection onto the first coordinate for arrows is an isomorphism of categories. □

4. Derived categories

Our derived category will be isomorphic to that in [11]. But like Tilson in [21], we will be more interested in the unfactored derived category which is in some sense, which can be made more precise [21], an adjoint [15] to the semidirect product, at least when the relational morphism is a quotient one (see below for definitions).

A *relational morphism* of categories $\varphi : C \to D$ is a relation so that the graph of the relation, $\#\varphi \subseteq C \times D$, is a subcategory which projects to C as a quotient map. If one is dealing with compact categories, we require $\#\varphi$ to be closed as well. The relational morphism is called a *division* if the projection to D is faithful. It is called a *quotient relational morphism* if the projection to D is also a quotient morphism. An analogous definition can be made for semigroupoids. Note, a relational morphism φ consists, by abuse of notation, of a function φ on vertices and a relation φ on edges which is fully defined when restricted to each hom set.

Let $\varphi : C \to D$ be a relational morphism. We define the *unfactored derived category* W_φ by

- $V(W_\varphi) = \{((c_1, c_2), \tilde{f}) \mid c_1, c_2 \in V(C),\ \exists g \in C(c_1, c_2) \text{ such that } \tilde{f} \in g\varphi\}$.
- $E(W_\varphi) = \{(((c_1, c_2), \tilde{f}), (g, f)) \mid ((c_1, c_2), \tilde{f}) \in V(W_\varphi)\ g\alpha = c_2,\ (g, f) \in E(\#\varphi)\}$.
- $(((c_1, c_2), \tilde{f}), (g, f))\alpha = ((c_1, c_2), \tilde{f})$.
- $(((c_1, c_2), \tilde{f}), (g, f))\omega = ((c_1, g\omega), \tilde{f}f)$.
- Multiplication is given by

$$((c_1, c_2), \tilde{f}) \xrightarrow{(g,f)} ((c_1, g\omega), \tilde{f}f) \xrightarrow{(g',f')} ((c_1, g'\omega), \tilde{f}ff')$$
$$= ((c_1, c_2), \tilde{f}) \xrightarrow{(gg',ff')} ((c_1, g'\omega), \tilde{f}ff'),$$

  where all the arrows on the left-hand side are defined.
- Local identities are the arrows $(((c_1, c_2), \tilde{f}), (1_{c_2}, 1_{c_2\varphi}))$.

It is easy to see, this gives a well-defined category, equivalent to the standard definition if C and D are monoids.

We define

$$(((c_1,c_2),\tilde{f}),(g,f)) \equiv (((c_1,c_2),\tilde{f}),(g',f')),$$

where $\tilde{f}f=\tilde{f}f'$, if $\cdot g=\cdot g'$ on $C(c_1,c_2)\cap\tilde{f}\varphi^{-1}$. It is easy to check that this is a congruence. Also note that no two arrows with initial vertex $((c,c),1_{c\varphi})$ can be identified. Then the derived category D_φ is defined by $D_\varphi = W_\varphi/\equiv$. We use brackets to denote equivalence classes of arrows.

For a semigroupoid S, one defines S^c to be S with the addition of identity arrows at each vertex which does not have one. One then defines, given a relational morphism $\varphi : S\to T$ of semigroupoids, a relational morphism of categories $\varphi^c : S^c \to T^c$ by adding the edges $(1_s, 1_{s\varphi})$ to $\#\varphi$. We define W_φ to be the ideal of W_{φ^c} consisting of all the vertices, but where we require the second coordinate of the edges to actually be in $E(\#\varphi)$. D_φ is the similarly defined ideal of D_{φ^c} and is a quotient of W_φ.

Note, if φ is a relational morphism of compact, or profinite semigroupoids C, D, then W_φ is also compact or profinite respectively. Furthermore, D_φ, being a quotient, will be compact in the quotient topology. But if C and D are profinite, it does not seem D_φ will in general be profinite. This is one reason why we are more interested in W_φ.

The following lemma, proved by Tilson [21] for the monoid case, will be of utmost importance in all that follows as it relates both derived categories, and in fact, says that they contain essentially the same information.

Lemma 4.1. *Let $\eta : W_\varphi \twoheadrightarrow D_\varphi$ be the quotient map and suppose we have a relational morphism $\rho : W_\varphi \to D$ as in*

$$\begin{tikzcd} W_\varphi \arrow[r, "\rho"] \arrow[d, two heads, "\eta"'] & D \\ D_\varphi \arrow[ur, "\psi"] & \end{tikzcd}$$

Then $\psi=\eta^{-1}\rho$ is a division if and only if ρ is injective on each edge set

$$W_\varphi(((c,c),1_{c\varphi}),((c,c'),f)).$$

Such sets are called core hom sets.

Proof. Since arrows emanating from $((c,c),1_{c\varphi})$ are never identified, if ψ is a division, ρ has the desired property. Conversely, suppose ψ is injective on core hom sets. Suppose furthermore, $(((c_1,c_2),\tilde{f}),(g,f))$, $(((c_1,c_2),\tilde{f}),(g',f'))$, are coterminal in W_φ and ρ-relate to an arrow $h\in D$. First note $\tilde{f}f=\tilde{f}f'$. We show that these arrows are identified in D_φ. Suppose $\hat{g}\in C(c_1,c_2)\cap\tilde{f}\varphi^{-1}$. Then $(((c_1,c_1),1_{c_1\varphi}),(\hat{g},\tilde{f}))$

is an arrow of W_φ composable with $(((c_1,c_2),\tilde{f}),(g,f))$ and $(((c_1,c_2),\tilde{f}),(g',f'))$. Since ρ is fully defined, there exists h' which ρ-relates to $(((c_1,c_1),1_{c_1\varphi}),(\hat{g},f'))$ and is necessarily composable with h. Hence since ρ is a relational morphism, we have that

$$(((c_1,c_1),1_{c_1\varphi}),(\hat{g},\tilde{f}))(((c_1,c_2),\tilde{f}),(g,f)) = (((c_1,c_1),1_{c_1\varphi}),(\hat{g}g,\tilde{f}f))$$

$$(((c_1,c_1),1_{c_1\varphi}),(\hat{g},\tilde{f}))(((c_1,c_2),\tilde{f}),(g',f')) = (((c_1,c_1),1_{c_1\varphi}),(\hat{g}g',\tilde{f}f'))$$

both ρ-relate to $h'h$. But since $\tilde{f}f = \tilde{f}f'$ and ρ is injective on arrows from $((c_1,c_1), 1_{c_1\varphi})$, we see $\hat{g}g = \hat{g}g'$ as desired. $\square$

Core hom sets will play an important role in the theory. They will be the core hom sets with respect to a certain semidirect product which we will study.

We then have the following proposition, most of which appears in [11], though the above lemma will considerably simplify the proof. We use the notation $C < D$ to say C divides D.

Proposition 4.2. *Let $\phi : C \to D$, $\theta : D \to E$, $\psi : C \to E$ be relational morphisms of categories (semigroupoids). Then we have a, not necessarily commutative, diagram*

$$\begin{tikzcd} C \arrow[r, "\phi"] \arrow[dr, "\psi"'] & D \arrow[d, "\theta"] \\ & E. \end{tikzcd}$$

1. *Suppose $\psi = \phi\theta$ and θ is a division, Then $D_\psi < D_\phi$.*
2. *Suppose $\phi\theta \subseteq \psi$ and θ is a morphism. Then $D_\phi < D_\psi$.*
3. *Suppose $\psi = \phi\theta$ and ϕ is a division. Then $D_\psi < D_\theta$.*
4. *Let φ_i be a collection of relational morphisms. Then $W_{\prod \varphi_i} \simeq \prod W_{\varphi_i}$ and $D_{\prod \varphi_i} \cong \prod D_{\varphi_i}$.*

Proof. The last statement is straight forward to verify. We check the rest.

1. We will define $\rho : W_\psi \to D_\phi$. First we define ρ on objects as follows; let $((c_1,c_2),\tilde{f})$ be an object of W_ψ. By assumption there exists $g : c_1 \to c_2$ which ψ relates to $\tilde{f}$. Hence by assumption, there is a unique arrow $\hat{f} : c_1\phi \to c_2\phi$ such that $\hat{f} \in g\phi \cap \tilde{f}\theta^{-1}$. So define $((c_1,c_2),\tilde{f})\rho = ((c_1,c_2),\hat{f})$. In the semigroupoid case, we have all arrows $((c,c),1_{c\psi})$ sent to $((c,c),1_{c\phi})$ (this is automatic in the category case). On arrows, we define $(((c_1,c_2),\tilde{f}),(g,f))\rho = [((c_1,c_2),\hat{f}),(g,\bar{f})]$ with $\hat{f}$ as above and where $\bar{f}$ is the unique arrow such that $\bar{f} \in g\phi \cap f\theta^{-1}$.

This map is actually a morphism. It sends identities to identities since $1_{c\phi} \in 1_c\phi \cap 1_{c\psi}\theta^{-1}$. It is a morphism since if f, f' are composable, (g,f), $(g',f') \in \#\psi$, $\bar{f} \in g\phi \cap f\theta^{-1}$, and $\bar{f'} \in g'\phi \cap f'\theta^{-1}$, then $(\bar{f})\bar{f'} \in gg'\phi \cap ff'\theta^{-1}$.

We also have that ρ is injective on core hom sets. Consider an arrow

$$(((c,c),1_{c\psi}),(g,f)).$$

Then any coterminal arrow must be of the form $(((c,c),1_{c\psi}),(g',f))$. Hence we just need to show, g is uniquely determined by the image of the arrow. But $(((c,c),1_{c\psi}),(g,f))\rho = [((c,c),1_{c\phi}),(g,\bar{f})]$ and since such arrows are not identified, g is uniquely determined. Thus by Lemma 4.1, ρ induces a division.

2. We will define a morphism $\rho : W_\phi \to D_\psi$ on arrows, the effect on objects will thus be determined. Define $(((c_1,c_2),\tilde{f}),(g,f))\rho = [((c_1,c_2),\tilde{f}\theta),(g,f\theta)]$. This is well defined since $\phi\theta \subseteq \psi$. In the semigroupoid case, we send objects $((c,c),1_{c\phi})$ to $((c,c),1_{c\psi})$. This is clearly a morphism. Also $(((c,c),1_{c\phi}),(g,f))\rho = [((c,c),1_{c\psi}),(g,f\theta)]$, so the morphism is once again injective on core hom sets.

3. We define $\rho : W_\psi \to D_\theta$ on objects by $((c_1,c_2),\tilde{f})\rho = ((c_1\phi,c_2\phi),\tilde{f})$. This is well defined since $\psi = \phi\theta$. On arrows, define $(((c_1,c_2),\tilde{f}),(g,f))\rho = \{[((c_1\phi,c_2\phi),\tilde{f}),(h,f)] \mid h \in g\phi, f \in h\theta\}$. Then φ is fully defined since $\psi = \phi\theta$ implies $g\phi \cap f\theta^{-1} \neq \emptyset$. Also, such $(h,f) \in \#\psi$ since $\psi = \theta\phi$. If g and f are identity arrows, we can choose h to be the appropriate identity arrow. Clearly ρ is multiplicative since ϕ and θ are. Thus ρ is a relational morphism. Suppose two coterminal arrows

$$(((c,c),1_{c\psi}),(g,f)),(((c,c),1_{c\psi}),(g',f))$$

ρ-relate to $[((c\phi,c\phi),1_{c\phi}),(h,f)]$. Then h ϕ-relates to both g and g'. Since ϕ is a division, $g = g'$. So ρ is injective on core hom sets. $\square$

Lemma 4.3. *Let $C *_{\psi,\rho} D$ be a semidirect product. Then for $\pi : C * D \to D$ the projection, $D_\pi < C$.*

Proof. As usual we write C additively. We define a morphism $\rho : W_\pi \to C$ on arrows, the effect on objects thus determined. Let

$$(((d_1,d_2),\tilde{f}),((g,f),f))\rho = \tilde{f}g,$$

where in the semigroupoid case, we have new identities act as an identity. This is well defined since $\tilde{f} : d_1 \to d_2$, $f : d_2 \to d_3$, and $g : d_2\psi \to fd_3\psi$ implies $\tilde{f}g$ is defined. Also $(((d_1,d_2),\tilde{f}),\ ((0_{d_2\psi},1_{d_2}),1_{d_2}))\rho = \tilde{f}0_{d_2\psi} = 0_{\tilde{f}d_2\psi}$. It is multiplicative since given composable arrows,

$$(((d_1,d_2),\tilde{f}),((g,f),f)),(((d_2,d_3),\tilde{f}f),((g',f'),f')),$$

we have

$$
\begin{aligned}
&((((d_1,d_2),\tilde{f}),((g,f),f)),(((d_2,d_3),\tilde{f}f),((g',f'),f')))\rho\\
&\quad =(((d_1,d_2),\tilde{f}),((g+fg',ff'),ff'))\rho\\
&\quad =\tilde{f}(g+fg')=\tilde{f}g+\tilde{f}fg'\\
&\quad =((((d_1,d_2),\tilde{f}),((g,f),f)))\rho((((d_2,d_3),\tilde{f}f),((g',f'),f')))\rho.
\end{aligned}
$$

Finally $((((d,d),1_d),((g,f),f)))\rho =: 1_d g = g$ so this morphism is injective on core hom sets. □

The following is then one-half of a derived category theorem on the category level.

Corollary 4.4. *Suppose we have commutative diagram of relational morphisms*

$$
\begin{tikzcd}
C \arrow[r, "\theta"] \arrow[dr, "\varphi"'] & D * E \arrow[d, "\pi"] \\
 & E
\end{tikzcd}
$$

with θ a division. Then $D_\varphi < D$.

Proof. By Proposition 4.2 and the above lemma we have $D_\varphi < D_\pi < D$. □

The other half of a derived category theorem makes sense on the category level only when $\varphi : C \to D$ is a quotient relational morphism.

Theorem 4.5. *Let $\varphi : C \to D$ be a quotient relational morphism. Then $C < W_\varphi * D$. If φ is a morphism, then C is a subcategory of $W_\varphi * D$.*

Proof. Without loss of generality, we may assume C and D have the same objects and φ is the identity on objects. Then we can identify the $V(W_\phi)$ with $E(D)$ since the relational morphism is identity on objects and onto on hom sets. Define a left action of D on W_φ by left multiplication on the edge coordinate when defined, that is $h(f',(g,f)) = (hf',(g,f))$ when hf' is defined. The base point map takes an object $d \in V(D)$ to 1_d. Define $\psi : C \to W_\varphi * D$ by the identity on objects and on arrows $f \in C(d_1,d_2)$ by $f\psi = \{((1_{d_1},(f,g)),g) \mid g \in f\varphi\}$. This is clearly fully defined since φ is. Identities clearly relate to identities and it is clearly injective. We see ψ is multiplicative since if f, f' are composable and g, g' are in $f\varphi$, $f'\varphi$ respectively (and hence are composable), $g: d_1 \to d_2$, $g': d_2 \to d_3$, then

$$((1_{d_1},(f,g)),g)((1_{d_2},(f',g')),g') = ((1_{d_1},(f,g)),(g,(f',g')),gg')$$
$$= ((1_{d_1},(ff',gg')),gg').$$

Also note, this is a morphism if φ is a morphism. $\square$

This theorem should reconcile our two definitions of core hom sets. One could in fact show that this is in some sense the "universal" relational morphism of C into a semidirect product with D, as in [21] for the monoid case. We also have the following useful fact.

Proposition 4.6. *Let $\varphi: C \to D$ be a relational morphism of categories. Then φ is a division if and only if D_φ is a trivial category.*

Proof. If φ is injective, then the identifications force coterminal arrows to be equal. Conversely, if D_φ is trivial, suppose (f,g), $(f',g) \in \#\varphi$ and $g: c_1\varphi \to c_2\varphi, f,g \in C(c_1,c_2)$. Then using that $[((c_1,c_1),1_{c_1\varphi}),(f,g)] = [((c_1,c_1),1_{c_1\varphi}),(f',g)]$ and that core hom set arrows have no identifications, we see $f = f'$, so φ is injective. $\square$

5. Varieties and pseudovarieties

We will now define two apparently different (pseudo) varieties of categories, one in terms of the semidirect product and the other in terms of the derived category. The derived category theorem on the (pseudo) variety level will then say that these two are equal. Our approach will be to use free objects, not wreath products. We will proceed to show our semidirect product operator commutes with the operation of taking the global of a (pseudo) variety. Then using our derived category theorem, we will calculate various globals of pseudovarieties.

Recall, a *variety* of categories is a collection of categories closed under products and divisions while a *pseudovariety* of categories is a collection of finite categories closed under finite products and divisions. There are analogous definitions for semigroupoids except, one requires the above collections be closed under coproducts as well, see [6, 20]. We also note that one has analogous definitions for a (pseudo) variety of monoids and semigroups. If $\mathbf{V}$ is a (pseudo) variety of monoids (semigroups), then $g\mathbf{V}$ represents the pseudovariety of categories (semigroupoids) generated by members of $\mathbf{V}$, viewed as one object categories (semigroupoids) and is called the global of $\mathbf{V}$.

Let $\mathbf{V}$, $\mathbf{W}$ be (pseudo) varieties of categories. Let

$$\mathbf{V} * \mathbf{W} = \{C \mid C < V * W, V \in \mathbf{V}, W \in \mathbf{W}\}.$$

Proposition 3.1 easily implies this is a (pseudo) variety. Proposition 3.4 shows it contains $\mathbf{V}$ and $\mathbf{W}$ and hence their join $\mathbf{V} \vee \mathbf{W}$.

We now define a (pseudo) variety $\mathbf{V} \circledast \mathbf{W} = \{C \mid \exists \varphi : C \to W,\ W \in \mathbf{W}$ a relational morphism such that $D_\varphi \in \mathbf{V}\}$. It follows from Proposition 4.2 that this is a (pseudo) variety.

Corollary 5.1. $\mathbf{V} * \mathbf{W} \subseteq \mathbf{W} \circledast \mathbf{W}$.

Proof. Immediate from Corollary 4.4. $\square$

One of the main results of this paper is the derived category theorem which states that $\mathbf{V} * \mathbf{W} = \mathbf{V} \circledast \mathbf{W}$. To do this, we will first construct free objects for $\mathbf{V} * \mathbf{W}$.

6. Free objects for semidirect products

Here we generalize standard results on free objects and free profinite objects for semidirect products, see [1, 6].

We first construct the free objects in the variety setting. We will then be able to obtain all our desired results using that pseudovarieties are directed unions of the finite traces of locally finite varieties. We will also give a characterization of the free profinite objects. Since we will not use this characterization and since the proof is a straight forward generalization of the semigroup case in [6], we will omit the details (which essentially consists of the same trivial verifications that certain maps are continuous, as in the monoid case, and, verifying that they are inverse limits of the free objects for the locally finite pseudovarieties, also identical to the semigroup case).

We first ask the reader to recall from Section 2 the definition of the Cayley graph of a category C with respect to a generating graph X. For a variety $\mathbf{V}$ we will denote the relatively free category on a graph Y by $F_{\mathbf{V}}(Y)$. We write Y^* for the free category on Y.

Let X be a graph and let $\mathbf{V}$, $\mathbf{W}$ be varieties of categories. We will show the left action of $F_{\mathbf{W}}(X)$ on $\Gamma_X(F_{\mathbf{W}}(X))$ induces a left action of $F_{\mathbf{W}}(X)$ on $F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))$. First we need some lemmas on free categories and semigroupoids.

Lemma 6.1. *Let $\mathbf{V}$ be a variety of categories (semigroupoids), X a graph and Y a union of connected components of X. Then the inclusion map induces a inclusion $F_{\mathbf{V}}(Y) \subseteq F_{\mathbf{V}}(X)$.*

Proof. We will handle both the category and semigroupoid case by using that varieties are closed under coproducts (one can give a simpler argument if one ignores the semigroupoid case). First note that we have an inclusion on objects. We write $\coprod$ for coproduct (note for categories and semigroupoids, the coproduct is disjoint union). Consider $C = F_{\mathbf{V}}(Y) \coprod T$ where T is the trivial one vertex category. Then $C \in \mathbf{V}$. Map X to C by mapping Y via the canonical map of Y into $F_{\mathbf{V}}(Y)$ and mapping all other vertices and arrows to T. Since Y is a union of connected components, this is a well-defined graph morphism. By considering also the inclusion of $F_{\mathbf{V}}(Y)$ into C, it follows the induced map $F_{\mathbf{V}}(Y)$ into $F_{\mathbf{V}}(X)$ cannot identify arrows. $\square$

A lemma of similar nature which we shall need later is the following.

Lemma 6.2. *Let* $\mathbf{V}$ *be a variety of monoids* (*semigroups*), X *a set*, Y *a subset. Then the inclusion* $Y \subset X$ *induces an inclusion* $F_{\mathbf{V}}(Y) \subseteq F_{\mathbf{V}}(X)$.

Proof. Map X to $F_{\mathbf{V}}(Y)$ by the identity on Y and arbitrarily on the rest of X. The same argument as above shows that the canonical map of $F_{\mathbf{V}}(Y)$ into $F_{\mathbf{V}}(X)$ must be injective. $\square$

We will write $\equiv_{\mathbf{V}}$ for the congruence on X^* which gives $F_{\mathbf{V}}(X)$. It is easy to see [20] that

$$\equiv_{\mathbf{V}} = \bigcap \{\equiv \mid X^*/\!\equiv \in \mathbf{V}\}.$$

We will moreover need the following, easily proved, proposition [20, Proposition 6.5]. If $\tau : C \to D$ is a morphism of categories, we write $\equiv_\tau$ for the associated congruence.

Proposition 6.3. *Let* S, T *be categories such that* $S < T$. *Let* $\tau : X^* \to S$ *be a morphism. Then there exists a morphism* $\tau' : X^* \to T$ *such that* $\equiv_{\tau'} \subseteq \equiv_\tau$.

We then have the following two consequences.

Corollary 6.4. *Let* $\mathbf{V}$ *be a variety of categories and* S *a set of elements of* $\mathbf{V}$ *such that every element of* $\mathbf{V}$ *is a divisor of an element of* S. *Then for any graph* X,

$$\equiv_{\mathbf{V}} = \bigcap \{\equiv_\tau \mid \tau : X^* \to C,\ C \in S\}.$$

Proof. Proposition 6.3 shows that such congruences form a dense subset of the lattice of all $\mathbf{V}$-congruences on X^*. The result follows. $\square$

We then have the following corollary which is done for free profinite objects in the pseudovariety case in [2].

Corollary 6.5. *Let* $\mathbf{V}$ *be a variety of monoids*, X *a graph. Then the natural faithful map of* X *into* $E(X)$ (*viewed as a one vertex graph*) *induces a faithful morphism of* $\varphi : F_{g\mathbf{V}}(X) \to F_{\mathbf{V}}(E(X))$.

Proof. The above corollary shows that $\equiv_{g\mathbf{V}}$ is the intersection of all congruences on X^* arising from morphisms into monoids in $\mathbf{V}$. But the image of X^* in any monoid is $E(X)$-generated and hence, we just need to intersect the congruences from morphisms to $E(X)$-generated monoids where the morphism sends edges of X to the corresponding generator. But all such morphisms factor through φ and thus $\equiv_\varphi = \equiv_{\mathbf{V}}$. $\square$

Consider an edge in the Cayley graph of the category $F_{\mathbf{W}}(X)$. We will call vertices of X objects and of $\Gamma_X(F_{\mathbf{W}}(X))$ vertices. It is easy to see that in $\Gamma_X(F_{\mathbf{W}}(X))$, both vertices of an edge have the same initial object of X. Hence, the initial object of the vertices is an invariant of paths and even undirected paths. Hence, the initial object of each vertex is an invariant of connected components of $\Gamma_X(F_{\mathbf{W}}(X))$. Let w be an arrow of $F_{\mathbf{W}}(X)$. Then $w\cdot$ is defined on the full subgraph whose vertices have initial object $w\omega$. Hence $Y = dom(w\cdot)$ is a union of connected components of $\Gamma_X(F_{\mathbf{W}}(X))$, in fact, it is a connected component since all such vertices are connected to $1_{w\omega}$. Now $w\cdot : Y \to \Gamma_X(F_{\mathbf{W}}(X))$ induces a morphism from Y^* to $F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))$ via the inclusion of $\Gamma_X(F_{\mathbf{W}}(X))$. Hence, we get an induced morphism $w\cdot$ from $F_{\mathbf{V}}(Y)$ into $F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))$. Also note Y^* is a full subcategory of $\Gamma_X(F_{\mathbf{W}}(X))^*$, since Y is a connected component of $\Gamma_X(F_{\mathbf{W}}(X))$. By Lemma 6.1, $F_{\mathbf{V}}(Y)$ is the subcategory of $F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))$ generated by Y and hence is full, since Y^* is full in $\Gamma_X(F_{\mathbf{W}}(X))^*$. Thus, we have defined a partial full endofunctor of $F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))$. Also by construction (as extensions of functors to free objects), these partial functors are uniquely determined by their action on their domains in $\Gamma_X(F_{\mathbf{W}}(X))$. It follows that the identity arrows of $F_{\mathbf{W}}(X)$ act as partial identities, the map ρ into $\mathrm{PFEnd_L}(F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X))))$ is a partial homomorphism (indeed if ww' is defined, then $w\rho w'\rho$ and $ww'\rho$ both have the same domain and on those edges of the domain from $\Gamma_X(F_{\mathbf{W}}(X))$ they agree). Furthermore, since the domain of $w\cdot$ is the full subcategory containing the vertices of $\Gamma_X(F_{\mathbf{W}}(X))$ whose initial object is $w\omega$, we see that the action has $\mathscr{L}$-consistency of domain. Finally, for each $w \in F_{\mathbf{W}}(X)$, the following diagram commutes

$$\begin{tikzcd}
Y \arrow[r, "w\cdot"] \arrow[d] & \Gamma_X(F_{\mathbf{W}}(X)) \arrow[d] \\
F_{\mathbf{V}}(Y) \arrow[r, "w\cdot"] & F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))
\end{tikzcd}$$

We choose a basepoint map ψ by sending $x \in V(X)$ to 1_x. Then if $w : x_1 \to x_2$, $wx_2\psi = w$ and hence is defined. Thus we have a well defined semidirect product

$$F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X))) * F_{\mathbf{W}}(X).$$

We map X in by identity on objects and on arrows by $x \mapsto ((1_{x\alpha}, x), x)$, call this map ρ. As usual, we will ignore the distinction between a graph morphism defined on X and the induced morphism defined on X^*.

Theorem 6.6. $F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X))) * F_{\mathbf{W}}(X) = F_{\mathbf{V}*\mathbf{W}}(X)$.

Proof. First note, $F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X))) * F_{\mathbf{W}}(X)$ is X-generated. Indeed for $y_1, y_2 \in V(X)$, an arrow in the corresponding hom set looks like (e, w), where $w : y_1 \to y_2 \in E(F_{\mathbf{W}}(X))$

and $e:1_{y_1}\to w\in F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))$. But any such e is the image of a path

$$1_{y_1}\xrightarrow{x_1}x_1\xrightarrow{x_2}\cdots\xrightarrow{x_{n-1}}x_1\ldots x_{n-1}\xrightarrow{x_n}w$$

$\Gamma_X(F_{\mathbf{W}}(X))^*$. But then it is easy to see, $(e,w)=((1_{y_1},x_1),x_1)\ldots((1_{x_1\ldots x_n\alpha},x_n),x_n)\rho$. By Corollary 6.4 we know that

$$\equiv_{\mathbf{V}*\mathbf{W}}=\bigcap\{\equiv_\tau \mid \tau:X^*\to V*W,\ V\in\mathbf{V},\ W\in\mathbf{W}\}.$$

Thus, we just need to show all such τ factor through $F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))*F_{\mathbf{W}}(X)$, or equivalently, we must show that given a map $\tau:X\to V*W$, $V\in\mathbf{V}$, $W\in\mathbf{W}$, there exists a map $\beta:F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))*F_{\mathbf{W}}(X)\to V*W$ such that $((1_{x\alpha},x),x)\beta=x\tau$ and is τ on objects.

For $x:x_1\to x_2\in X$ let $x\tau=(x\tau_1,x\tau_2)$. Let $\beta_2:F_{\mathbf{W}}(X)\to W$ be the unique extension of τ_2, so on objects $\tau_2=\tau$. For semigroupoids we take $1_{x_i}\beta_2=1_{x_i\tau}$.

Consider $\gamma:\Gamma_X(F_{\mathbf{W}}(X))\to V$ defined on arrows by $(w,x)\gamma=w\beta_2x\tau_1$. This is well defined since if $w:x_1\to x_2$ and $x:x_2\to x_3$, then $x\tau_1:x_2\tau_2\psi'\to x\tau_2(x_3\tau_2\psi')$, where ψ' is the basepoint map for $V*W$, and $w\beta_2:x_1\tau_2\to x_2\tau_2$. Hence on vertices, $w\gamma=w\beta_2(x_2\tau_2\psi')$ where $w:x_1\to x_2$. Now extend γ to a map $\beta_1:F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))\to V$.

We now show the β_i are compatible with the semidirect product actions. First note for vertices $x_i\in X$,

$$x_i\psi\beta_1=1_{x_i}\beta_2x_i\tau_2\psi'=x_i\tau_2\psi'=x_i\beta_2\psi'.$$

Secondly if $w\cdot$ is defined on a vertex w' of $\Gamma_X(F_{\mathbf{W}}(X))$,

$$(ww')\beta_1=(ww')\beta_2(ww')\omega\tau_2\psi'=w\beta_2w'\beta_2w'\omega\tau_2\psi'=w\beta_2w'\beta_1.$$

Finally, if $w:x_1\to x_2$ and (w',x) is in the domain of $w\cdot$, then

$$(w(w',x))\beta_1=(ww',x)\beta_1=(ww')\beta_2x\tau_1=w\beta_2w'\beta_2x\tau_1=w\beta_2(w',x)\beta_1.$$

Hence iterating on paths and using that $w\cdot$ is a partial full endofunctor, we see $(w\cdot e)\beta_1=w\beta_2e\beta_1$ for any $e\in F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))$. Thus, we see the β_i are compatible with the semidirect product actions. Hence, we get a well defined morphism $\beta:F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))*F_{\mathbf{W}}(X)\to V*W$ given by β_2 on objects and $\beta_1\times\beta_2$ on arrows. But $\beta_2=\tau$ on objects and

$$((1_{x\alpha},x),x)\beta=(1_x\beta_2x\tau_1,x\beta_2)=(x\tau_1,x\tau_2)=x\tau.$$

So β is the desired map. $\square$

Recall a variety $\mathbf{V}$ is called *locally finite* if finitely generated members are finite. It is well known that the variety generated by a single finite category, for instance, is locally finite, see [20].

Corollary 6.7. *If* $\mathbf{V}$ *and* $\mathbf{W}$ *are locally finite, so is* $\mathbf{V} * \mathbf{W}$. *Furthermore, if we write* $(\ )^F$ *for the operator which associates to a variety the pseudovariety of finite members (the finite trace), then if* $\mathbf{V}$ *and* $\mathbf{W}$ *are locally finite varieties,* $\mathbf{V}^F * \mathbf{W}^F = (\mathbf{V} * \mathbf{W})^F$.

Proof. The first statement is clear. The only non-trivial part of the second statement is that $(\mathbf{V}*\mathbf{W})^F \subseteq \mathbf{V}^F * \mathbf{W}^F$. But if $C \in \mathbf{V}*\mathbf{W}$ is finite, then it can be generated by a finite graph X and hence our description of the free objects shows that $F_{\mathbf{V}*\mathbf{W}}(X) \in \mathbf{V}^F * \mathbf{W}^F$ and thus so is C. $\square$

We denote the free pro-$\mathbf{V}$ object generated by a graph X for a pseudovariety $\mathbf{V}, \overline{\Omega}_X \mathbf{V}$, see [1] for more. The following theorem is proved from the above one in a similar manner to the semigroup analog in [6].

Theorem 6.8. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties and* X *a finite graph. Then* $\overline{\Omega}_X(\mathbf{V}*\mathbf{W}) = \overline{\Omega}_{\Gamma_X(\overline{\Omega}_X \mathbf{W})} * \overline{\Omega}_X \mathbf{W}$.

7. The derived category theorem

We are now ready to prove one of our main results, the derived category theorem. Already we have one half by Corollary 5.1 which states $\mathbf{V} * \mathbf{W} \subseteq \mathbf{V} \circledast \mathbf{W}$. We begin with a lemma.

Lemma 7.1. *Let* C *be a category generated by a graph* X *such that the map of* X *into* C *is a bijection on objects. Without loss of generality, we may assume* $V(C) = V(X)$. *Let* $\mathbf{W}$ *be a variety of categories,* $\varphi : C \to F_{\mathbf{W}}(X)$ *be the canonical quotient relational morphism*

$$\begin{tikzcd} X^* \arrow[r, two heads] \arrow[d, two heads] & F_{\mathbf{W}}(X) \\ C \arrow[ur, "\varphi"] & \end{tikzcd}$$

Then $C \in \mathbf{V} \circledast \mathbf{W}$ *if and only if* $D_\varphi \in V$.

Proof. The reverse direction is clear. Conversely, if $C \in \mathbf{V} \circledast \mathbf{W}$, then consider a relational morphism φ'

$$\begin{tikzcd} R \arrow[r] \arrow[d, two heads] & W \\ C \arrow[ur, "\varphi'"] & \end{tikzcd}$$

with $W \in \mathbf{W}$, $D_{\varphi'} \in V$. By lifting the edges of X to R and considering the subcategory so generated, we get a relational morphism

$$\begin{tikzcd} X^* \arrow[r] \arrow[d, two heads] & W \\ C \arrow[ur, "\varphi''"] & \end{tikzcd}$$

with $\varphi'' \subseteq \varphi'$ and hence a factorization

$$\begin{tikzcd} X^* \arrow[r, two heads] \arrow[d, two heads] & F_{\mathbf{W}}(X) \arrow[r, "\psi"] & W \\ C \arrow[ur, "\varphi"] \arrow[urr, "\varphi''"'] & & \end{tikzcd}$$

with ψ a morphism, $\varphi\psi = \varphi'' \subseteq \varphi'$. Hence by Proposition 4.2, $D_\varphi < D_{\varphi'}$. $\square$

We will now be under the standing assumption that C is generated by a graph X such that $V(X) = V(C)$ (where we map X in by the identity on objects) and φ is the canonical quotient relational morphism with $F_{\mathbf{W}}(X)$. Then as in Theorem 4.5 we may identify $V(W_\varphi)$ with $E(F_{\mathbf{W}}(X))$. It is easy then to see W_φ is generated by $\Gamma_X(F_{\mathbf{W}}(X))$ under the map defined on vertices by the identity map and on edges by

$$(w,x) \mapsto (w,(x,x)).$$

This follows precisely because φ is defined by sending generator to generator. Hence D_φ is also generated by $\Gamma_X(F_{\mathbf{W}}(X))$ under the composition of the above map with the quotient map $\eta : W_\varphi \twoheadrightarrow D_\varphi$. Let $\psi : W_\varphi \to F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))$ be the canonical quotient relational morphism (the one induced by sending generator to generator). Then $\psi' = \eta^{-1}\psi$ is the canonical quotient relational morphism of D_φ with $F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))$. But this quotient relational morphism is a division precisely when $(\psi')^{-1}$ is a quotient morphism, which is precisely when $D_\varphi \in \mathbf{V}$, since D_φ is $\Gamma_X(F_{\mathbf{W}}(X))$ generated. Hence, Lemmas 4.1 and 7.1 give

Lemma 7.2. *A category C as above is in $\mathbf{V} \circledast \mathbf{W}$ if and only if the canonical quotient morphism $\psi : W_\varphi \to F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))$ is injective on core hom sets.*

Theorem 7.3. *Let $\mathbf{V}$, $\mathbf{W}$ be varieties. Then $\mathbf{V} * \mathbf{W} = \mathbf{V} \circledast \mathbf{W}$.*

Proof. By Corollary 5.1, we just need to show $\mathbf{V} \circledast \mathbf{W} \subseteq \mathbf{V} * \mathbf{W}$. So suppose C as above is in $\mathbf{V} \circledast \mathbf{W}$. Then by Theorem 4.5, $C < W_\varphi * F_{\mathbf{W}}(X)$, where the action of $F_{\mathbf{W}}(X)$

on W_φ is the one induced by the left action of $F_{\mathbf{W}}(X)$ on $\Gamma_X(F_{\mathbf{W}}(X))$), namely left multiplication on vertices (when defined) and left multiplication on the first coordinate of an edge. The basepoint map chooses the identity arrow for each object of X. Thus the core hom sets of W_φ in both senses we have defined, are the same. Furthermore by the lemma above, if $\psi : W_\varphi \to F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X)))$ is the canonical quotient morphism, ψ is injective on the core. Thus we just need, by Proposition 3.3, to show that ψ is compatible with the semidirect product actions of $W_\varphi * F_{\mathbf{W}}(X)$ and $F_{\mathbf{V}}(\Gamma_X(F_{\mathbf{W}}(X))) * F_{\mathbf{W}}(X)$. But this follows immediately since both actions are induced by the left action of $F_{\mathbf{W}}(X)$ on $\Gamma_X(F_{\mathbf{W}}(X))^*$ and the basepoint maps are the same. Thus $C \in \mathbf{V} * \mathbf{W}$. $\square$

We will now show the above theorem holds for pseudovarieties. One could redo the above arguments using free profinite objects and checking continuity at the correct places, but it is technically easier to use the fact that any pseudovariety is a directed union of the finite traces of locally finite varieties. This follows easily from the fact that a single finite category generates a locally finite variety and in fact, one can easily choose the locally finite varieties to form a chain, see [8, 20].

Theorem 7.4. *Let* $\mathbf{V}, \mathbf{W}$ *be pseudovarieties. Then* $\mathbf{V} * \mathbf{W} = \mathbf{V} \circledast \mathbf{W}$.

Proof. Let $\mathbf{V}_1 \subseteq \mathbf{V}_2 \subseteq \cdots$, $\mathbf{W}_1 \subseteq \mathbf{W}_2 \subseteq \cdots$ be representations of $\mathbf{V}$ and $\mathbf{W}$, respectively, as ascending unions of the finite traces of locally finite varieties. As usual by Corollary 5.1, we just need to show that if $C \in \mathbf{V} \circledast \mathbf{W}$, then $C \in \mathbf{V} * \mathbf{W}$. Suppose there exists $W \in \mathbf{W}$ and $\varphi : C \to W$ a relational morphism such that $D_\varphi \in \mathbf{V}$. Choose i, j such that $W \in \mathbf{W}_j$ and $D_\varphi \in \mathbf{V}_i$. Then $C \in \mathbf{V}_i * \mathbf{W}_j$ on the variety level. Hence $C \in (\mathbf{V}_i * \mathbf{W}_j)^F = \mathbf{V}_i^F * \mathbf{W}_j^F$ by Corollary 6.7. Hence, $C \in \mathbf{V} * \mathbf{W}$. $\square$

8. Connection with the semidirect products of monoids

As usual, all the results of this section apply verbatim to semigroups, merely replacing monoid with semigroup and category with semigroupoid.

Recall that if $\mathbf{V}$ is a (pseudo) variety of monoids, we denote the (pseudo) variety of categories generated by $\mathbf{V}, g\mathbf{V}$. Then we have the following commutation results (which are truly the motivation for this paper).

Theorem 8.1. *Let* $\mathbf{V}, \mathbf{W}$ *be varieties of monoids. Then* $g(\mathbf{V} * \mathbf{W}) = g\mathbf{V} * g\mathbf{W}$.

Proof. Clearly $g(\mathbf{V} * \mathbf{W}) \subseteq g\mathbf{V} * g\mathbf{W}$. For the other direction, it suffices to show that for any graph X, we have that the category $F_{g\mathbf{V}}(\Gamma_X(F_{g\mathbf{W}}(X))) * F_{g\mathbf{W}}(X)$ maps faithfully into the monoid $F_{\mathbf{V}}(E(\Gamma_{E(X)}(F_{\mathbf{W}}(E(X))))) * F_{\mathbf{W}}(E(X))$. We already know by Corollary 6.5 that the natural map

$$\varphi : F_{g\mathbf{W}}(X) \to F_{\mathbf{W}}(E(X))$$

is faithful. We also know that the natural map

$$\tau : F_{g\mathbf{V}}(\Gamma_X(F_{g\mathbf{W}}(X))) \to F_{\mathbf{V}}(E(\Gamma_X(F_{g\mathbf{W}}(X))))$$

is faithful. In addition there is a map

$$\psi : F_{\mathbf{V}}(E(\Gamma_X(F_{g\mathbf{W}}(X)))) \to F_{\mathbf{V}}(E(\Gamma_{E(X)}(F_{\mathbf{W}}(E(X)))))$$

defined on generators by

$$(w,x) \to (w\varphi, x).$$

We show that the composite map $\tau\psi$ is faithful. To show that a morphism of categories is faithful, it suffices to restrict to connected components. But the connected components of $F_{g\mathbf{V}}(\Gamma_X(F_{g\mathbf{W}}(X)))$ correspond to vertices with the same initial object. Let C be such a connected component. Then by Lemmas 6.1 and 6.2, the image of C in $F_{\mathbf{V}}(E(\Gamma_X(F_{g\mathbf{W}}(X))))$ is $F_{\mathbf{V}}(E(C))$. We show that the restriction of ψ to the image of the $E(C)$ is an injective map to

$$E(\Gamma_{E(X)}(F_{\mathbf{W}}(E(X)))).$$

It will then follow ψ restricted to $F_{\mathbf{V}}(E(C))$ is injective by Lemma 6.2 and the claim will follow. Indeed for $(w,x) \in E(C)$, $(w,x)\psi = (w\varphi, x)$ determines x and hence the endpoint of w. But the initial point of w is the same for all elements of C. Since φ is faithful it follows w is determined.

But the maps φ and $\tau\psi$ are clearly compatible with the semidirect product actions since

$$(w(w',x))\tau\psi = ((ww'),x)\tau\psi,$$

$$((ww')\varphi, x) = w\varphi(w'\varphi, x),$$

so the result follows from Proposition 3.2. Note this morphism is in fact the induced morphism from the composition of the faithful map of X into $E(X)$ with the inclusion of $E(X)$ into $F_{g(\mathbf{V}*\mathbf{W})}(E(X))$. $\square$

Now as before, we prove the result for pseudovarieties from the above result.

Theorem 8.2. *Let* $\mathbf{V}$, $\mathbf{W}$ *be pseudovarieties of monoids. Then* $g(\mathbf{V} * \mathbf{W}) = g\mathbf{V} * g\mathbf{W}$.

Proof. As before clearly $g(\mathbf{V} * \mathbf{W}) \subseteq g\mathbf{V} * g\mathbf{W}$. Let $C \in g\mathbf{V} * g\mathbf{W}$. Then there exists locally finite varieties $\mathbf{V}_i, \mathbf{W}_j$ with $\mathbf{V}_i^F \subseteq \mathbf{V}, \mathbf{W}_j^F \subseteq \mathbf{W}$ such that $C \in g\mathbf{V}_i * g\mathbf{W}_j$. If C is generated by a finite graph X, the above proof shows $C < F_{\mathbf{V}_i}(E(\Gamma_{E(X)}(F_{\mathbf{W}_j}(E(X))))) * F_{\mathbf{W}_j}(E(X))$. But each of the two factors of this semidirect product are finite since $\mathbf{V}_i$ and $\mathbf{W}_j$ are locally finite. So we see, $C \in g(\mathbf{V}_i^F * \mathbf{W}_j^F) \subseteq g(\mathbf{V} * \mathbf{W})$. $\square$

Note that this sort of commutation does not work for the semidirect product of a pseudovariety of categories with a pseudovariety of monoids $\circledast$ introduced in [6] (we no longer need the $\circledast$ notation for our semidirect product since $*$ gives the same result). Recall $\ell\mathbf{V}$ is the (pseudo) variety of all categories whose local monoids are in $\mathbf{V}$. Let $\mathbf{J}$ be the pseudovariety of $\mathscr{J}$-trivial semigroups. A result of Knast [14] shows that $\mathbf{J}$ is not local. It is clear $\ell\mathbf{J} \circledast \mathbf{1} = \mathbf{J}$. But $\ell\mathbf{J} \subseteq \ell\mathbf{J} * g\mathbf{1}$ by Proposition 3.4. So

$$g(\ell\mathbf{J} \circledast \mathbf{1}) = g\mathbf{J} \subsetneq \ell\mathbf{J} \subseteq \ell\mathbf{J} * g\mathbf{1}.$$

Clearly any non-local pseudovariety could have been used.

It is announced in [3] that Teixeira has extended the results of [6] to the semidirect product of category (and semigroupoid) varieties as defined by Jones and Pustejovksy [11]. The quoted result, combined with the fact that $\mathbf{V}$ is hyperdecidable if and only if $g\mathbf{V}$ is, [3] shows the following, see [3, 5].

Corollary 8.3. *If $g\mathbf{V}$ has finite vertex rank or a recursively enumerable basis of computable pseudoidentities and $\mathbf{W}$ is hyperdecidable, then $g(\mathbf{V} * \mathbf{W})$ is decidable.*

We note that this announced extension of the main result of [6] can be derived in a straight forward manner from the proofs of Lemma 7.1 and Theorem 7.3, but where we instead use free profinite objects. For some consequences with respect to iterated semidirect products, see [5] which uses, amongst other things, Theorem 8.2 to decide a large class of semidirect products involving pseudovarieties such as $\mathbf{G}$, $\mathbf{J}$, $\mathbf{Ab}$, and $\mathbf{D}$.

9. Two-sided semidirect products

In this section we indicate the needed changes to make the above results go through for the two-sided semidirect product.

9.1. Double semidirect product

First if C is a category generated by a graph X, we define the *two-sided Cayley graph* with respect to $X, \Gamma_X(C)$, to have vertices $E(C) \times E(C)$ and edges $\{(f,x,g) \in E(C) \times E(X) \times E(C) \mid fx$ and xg are defined$\}$. The initial vertex of such an edge is (f,xg) the final vertex is (fx,g). There are natural commuting left and right partial actions of C on $\Gamma_X(C)$, defined on vertices by left multiplication in the first coordinate and right multiplication in the second coordinate respectively. As before, the actions are defined on full subgraphs.

More generally, we define a *double action* of a category D on a category C (analogous definitions are made for semigroupoids) to be commuting left and right actions, where a *right action* is dual to the left action defined above, and a right and left action *commute* if $(fc)g = f(cg)$ where f and g are arrows of D. To define the *double semidirect product* $C ** D$ one then needs a double action of D on C and a basepoint map $\psi : V(D) \to V(C)$ with the following property; if $g : d_1 \to d_2$ is an arrow of D, then $gd_2\psi$ and $d_1\psi g$ are defined. We then define

$$V(C ** D) = V(D)$$

and

$$(C ** D)(d_1, d_2) = \{(f, g) \mid g \in D(d_1, d_2),\ f \in C(d_1\psi g,\ gd_2\psi)\}.$$

As usual, we will write C additively. One defines multiplication by $(f,g)(f',g') = (fg' + gf')$. This multiplication is associative and $(0_{d\psi}, 1_d)$ is the local identity at d. The projection to D is a morphism which is bijective on objects. One then defines the core hom sets of C to be hom sets of the form $C(d_1\psi g, gd_2\psi)$ where $g : d_1 \to d_2$. The results of Section 3 then hold for double semidirect products with the obvious modifications.

9.2. Kernel category

We now define the analog of the derived category for two-sided semidirect products. We will be interested as before in both the factored and the unfactored versions of this category. Our kernel category is isomorphic to that of Jones and Pustejovksy [11]. Furthermore, we will see that the double semidirect product commutes with the global operator. Here we will just give the definitions, as the proofs are obtained by modifying the monoid theoretic proofs in [16, 21] in exactly the same way that they are modified in the one-sided case above.

If $\varphi : C \to D$ is a relational morphism of categories we define the *unfactored kernel category* W_φ by

$$V(W_\varphi) = \{(((c_1, c_2), g), (g', (c_2, c_3))) \mid g \in C(c_1, c_2)\varphi,\ g' \in C(c_2, c_3)\varphi\}$$

and

$$E(W_\varphi) = \{(((c_1, c_2), g), (f, \hat{g}), (g', (c_3, c_4))) \mid g \in C(c_1, c_2)\varphi,\ g' \in C(c_3, c_4)\varphi,$$
$$(f, \hat{g}) \in \#\varphi,\ f \in C(c_2, c_3)\}.$$

The initial vertex of such an arrow is $(((c_1, c_2), g), (\hat{g}g', (c_2, c_4)))$ and the terminal vertex $(((c_1, c_3), g\hat{g}), (g', (c_3, c_4)))$. Multiplication is defined as usual by

$$(((c_1, c_2), g), (f, \hat{g}), (\tilde{g}g', (c_3, c_5)))(((c_1, c_3), g\hat{g})(f', \tilde{g}), (g', (c_4, c_5)))$$
$$= (((c_1, c_2), g), (ff', \hat{g}\tilde{g}), (g', (c_4, c_5))).$$

The local identities are as usual $(((c_1, c_2), g), (1_{c_2}, 1_{c_2\varphi}), (g', (c_2, c_3)))$. We define the core hom sets to be those of the form

$$W_\varphi((((c_1, c_1), 1_{c_1\varphi}), (g, (c_1, c_2))), (((c_1, c_2), g), (1_{c_2\varphi}, (c_2, c_2)))).$$

We define the *kernel category* K_φ to be W_φ modulo the congruence

$$(((c_1,c_2),g),(f,\hat{g}),(g',(c_3,c_4))) \equiv (((c_1,c_2),g),(f',\tilde{g}),(g',(c_3,c_4))),$$

where $g\hat{g}=g\tilde{g}, \hat{g}g'=\tilde{g}g'$, if for all $a\in C(c_1,c_2)\cap g\varphi^{-1}$, $b\in C(c_3,c_4)\cap g'\varphi^{-1}$, $afb=af'b$. One easily checks this is a congruence.

Analogously to the one-sided case, we can now make two definitions of $\mathbf{V}**\mathbf{W}$ where $\mathbf{V}$ and $\mathbf{W}$ are (pseudo) varieties of categories. But it is not hard to verify that all the theorems in the one-sided case go through for the two-sided case, where throughout we replace D_φ with $K_\varphi, *$ with $**$, one-sided Cayley graphs with two-sided Cayley graphs, and we use the new definition of core hom sets. Hence if we define $\mathbf{V}**\mathbf{W}$ to be the (pseudo) variety generated by double semidirect products of categories in $\mathbf{V}$ and $\mathbf{W}$, we then have,

Theorem 9.1 (Kernel theorem). $\mathbf{V}**\mathbf{W}$ *consists of those categories C with a relational morphism into* $\mathbf{W}$ *such that the kernel is in* $\mathbf{V}$.

We also get

Theorem 9.2. $g\mathbf{V}**g\mathbf{W}=g(\mathbf{V}**\mathbf{W})$.

There is an analogous description of free (profinite) objects, but we must use the two-sided Cayley graph in place of the Cayley graph.

10. Applications

Our first application is that Theorem 9.2 and the one-sided version are needed for the results in [12], including that $\mathbf{DS}$ is local. Recall that $\mathbf{DS}$ is the pseudovariety of semigroups such that regular $\mathscr{J}$-classes form a semigroup.

One of the motivating drives of this research was to prove locality of certain pseudovarieties of the form $\mathbf{V}*\mathbf{G}$. Indeed one can use the above results to prove $\mathbf{V}*\mathbf{G}$ is local if $\mathbf{V}$ is a monoidal pseudovariety of bands. However, roughly the same ideas show that for $\mathbf{V}$ a pseudovariety of semigroups, $\mathbf{EV}$, the pseudovariety of semigroups whose idempotents generate a subsemigroup in $\mathbf{V}$, is always local so long as $\mathbf{V}$ contains a non-trivial semilattice.

Proposition 10.1. *If* $\mathbf{V}*\mathbf{G}=\mathbf{EV}$ *and* $\mathbf{V}$ *is a monoidal pseudovariety of bands, then* $\mathbf{V}*\mathbf{G}$ *is local.*

Proof. Since any monoidal pseudovariety of bands is local [13], a category C is in $g(\mathbf{V}*\mathbf{G})=g\mathbf{V}*g\mathbf{W}$ precisely when for each local monoid, the loops which relate to 1 for every relational morphism of C with a finite group, form a monoid in $\mathbf{V}$. But in [3] these are shown to be precisely loops whose image in the consolidation of C are type II (see [20] for the definition of the consolidation and [9] for more on type II).

But C being locally in $\mathbf{V} * \mathbf{G}$, means the local monoids have idempotents forming a band in $\mathbf{V}$. Hence we see that in the consolidation of C, the idempotents form a band and thus weak conjugation gives no new type II elements. So for each local monoid, the loops which relate to 1 in every relational morphism of C with a finite group are precisely the idempotents, which by assumption form a monoid in $\mathbf{V}$. $\square$

The more general result has a similar flavor but does not use the derived category. Recall, B_2 is the five element, aperiodic Brandt semigroup. Recall, the consolidation of a semigroupoid S is the semigroup S^{cd} formed by taking $E(S)$ with an additional zero and declaring that the product of non-composable edges be zero, while the other compositions remain the same. When we are dealing with monoids, we add identities to both B_2 and the consolidation.

Proposition 10.2. *Let* $\mathbf{V}$ *be a pseudovariety of semigroups containing a non-trivial semi-lattice. Then* $\mathbf{EV}$ *is local.*

Proof. First note $B_2 \in \mathbf{EV}$, since $\mathbf{SL} \subseteq \mathbf{V}$, and hence for a semigroupoid $C, C \in g\mathbf{EV}$ if and only if the consolidation C^{cd} is in $\mathbf{EV}$, see [20]. Let $\pi = \rho$ be a pseudoidentity satisfied by $\mathbf{V}$. Then π and ρ have the same content, that is use the same letters, since $\mathbf{SL} \subseteq \mathbf{V}$. Thus since the idempotents of C^{cd} are 0 and the idempotents of the local semigroups of C, when we evaluate both sides in $E(C^{cd})$, the subsemigroup generated by the idempotents of C^{cd}, we either get 0 or we are really evaluating the pseudoidentity in the subsemigroup generated by the idempotents of a local semigroup of C. Hence if C is locally in $\mathbf{EV}$, then it is globally as well. $\square$

Thus by the Delay Theorem [20],

Corollary 10.3. $\mathbf{EV} * \mathbf{D} = \mathbf{LEV}$ *where* $\mathbf{V}$ *is a pseudovariety of monoids, not entirely consisting of groups.*

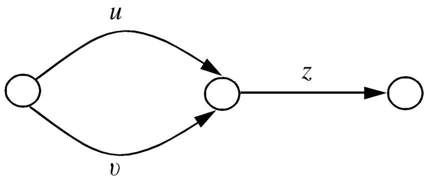
A particular case of this is the well known result $\mathbf{ER} * \mathbf{D} = \mathbf{LER}$ [8]. A similar result is the following.

Proposition 10.4. *If* $\mathbf{H}$ *is a pseudovariety of groups and* $\overline{\mathbf{H}}$ *is the pseudovariety of all semigroups whose subgroups are in* $\mathbf{H}$, *then* $\overline{\mathbf{H}}$ *is local.*

Proof. First note, since B_2 is aperiodic, to check membership in $g\overline{\mathbf{H}}$, we just need to check whether the subgroups of the consolidation of a semigroupoid are in $\mathbf{H}$. But it is easy to see that for a semigroupoid C, every non-trivial subgroup of C^{cd} is a subgroup of some local semigroup of C. The result follows. $\square$

We also have

Lemma 10.5. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties containing* B_2. *Then* $g(\mathbf{V} \cap \mathbf{W}) = g\mathbf{V} \cap g\mathbf{W}$.



$uz = vz$

Fig. 1. Pseudoidentity for $g\mathbf{RZ}$.

Proof. This is obvious since both sides are asking that the consolidation of a semigroupoid be in $\mathbf{V} \cap \mathbf{W}$. $\square$

Corollary 10.6. *If* $\mathbf{V}$ *and* $\mathbf{W}$ *are local pseudovarieties containing* B_2, *then* $\mathbf{V} \cap \mathbf{W}$ *is local.*

Proof. This follows from the above and that $\ell(\mathbf{V} \cap \mathbf{W}) = \ell\mathbf{V} \cap \ell\mathbf{W}$. $\square$

Thus the reader can combine the above propositions to obtain many instances of local pseudovarieties. For instance we recover the result of [4] $(\mathbf{ER} \cap \overline{\mathbf{H}}) * \mathbf{D} = \mathbf{L}(\mathbf{ER} \cap \overline{\mathbf{H}})$ for any pseudovariety of groups $\mathbf{H}$ and in fact more generally we have,

Corollary 10.7. *If* $\mathbf{V}$ *is a pseudovariety of monoids, not entirely consisting of groups and,* $\mathbf{H}$ *is a pseudovariety of groups, then* $(\mathbf{EV} \cap \overline{\mathbf{H}}) * \mathbf{D} = \mathbf{L}(\mathbf{EV} \cap \overline{\mathbf{H}})$.

As another application, we construct a basis of pseudoidentities for the global of the pseudovarieties of right simple $\mathbf{RS}$, left simple $\mathbf{LS}$, and completely simple $\mathbf{CS}$ semigroups, as well as for $\mathbf{N} \vee \mathbf{G}$. It should be clear how to get more examples from these methods. We first write a basis for $\mathbf{RZ}$, the pseudovariety of right zero semigroups. The identity of Fig. 1, is clearly satisfied by $g\mathbf{RZ}$. But for a semigroupoid C generated by a graph X, it is clear the canonical relational morphism with the free right zero semigroup on $E(X)$ is injective if C satisfies identity of Fig. 1. Also note, this pseudovariety is not local. Indeed take the free semigroupoid on the above graph. This semigroupoid does not satisfy the identity of Fig. 1, but is locally in $\mathbf{RZ}$.

Proposition 10.8. *The pseudoidentity of Fig.* 2 *defines* $g\mathbf{RS}$.

Proof. Clearly $\mathbf{RS}$ satisfies this pseudoidentity. For the converse we first note

$$\mathbf{RS} = \mathbf{RZ} * \mathbf{G} = \mathbf{RZ} \vee \mathbf{G},$$

so $g\mathbf{RS} = g\mathbf{RZ} * g\mathbf{G}$. Suppose C is a semigroupoid, then construct a relational morphism $\varphi : C \to G$ with a finite group G that gets correct $\mathbf{G}$-pointlikes, that is the intersection of a hom set of C with the inverse image of an element of G is a $\mathbf{G}$-pointlike set. Such a finite group always exists (see [3] or [19] for more on pointlikes of semigroupoids).

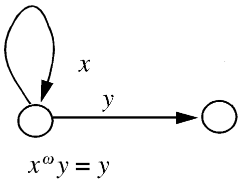


Fig. 2. Pseudoidentity for $g\mathbf{RS}$.

The description of pointlikes with respect to $\mathbf{G}$ is the same for semigroupoids as it is for semigroups (again see [3] or [19]). We say f is a weak inverse of g if $fgf = f$. Then the pointlike sets, see [7, 9], are of the form $w_1^{\varepsilon_1} \cdots w_n^{\varepsilon_n}$ where ε_i is 1 or -1, w^1 is the singleton $\{w\}$, and w^{-1} is the set of weak inverses of w. Suppose we map in the graph from Fig. 1 into D_φ. Let $u \mapsto (c, g), v \mapsto (c', g')$, and $z \mapsto (c'', g'')$. Then since G is a group, we see $g = g'$ and so $\{c, c'\}$ is a pointlike set (they must be in the same hom set since (c, g) and (c, g') are in the same hom set of D_φ). We now show that if C satisfies the pseudoidentity of Fig. 2, and if f and h are weak inverses of an arrow k, then $fx = hx$ for all composable x (note f and h, as weak inverses, of k must be coterminal). Then by induction, it easily follows from the above and our description of pointlikes for $\mathbf{G}$ that if f and h are $\mathbf{G}$-pointlike, then $fx = hx$ for all composable edges x. This will then imply that D_φ satisfies $uz = vz$ and the result follows. So suppose f and h are weak inverses of k. Then fk and kh are idempotent. So by the identity we have assumed for C,

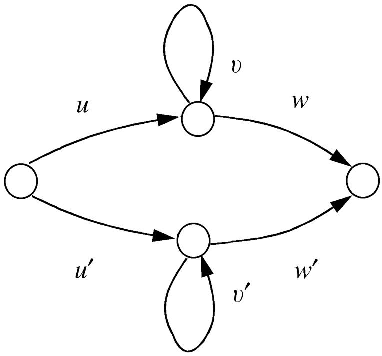
$$hx = (fk)hx = f(kh)x = fx. \qquad \square$$

By taking the free semigroupoid on the above graph with the relation $x^2 = x$, we obtain a semigroupoid locally in $\mathbf{RS}$ but not globally. Dual pseudoidentities define $\mathbf{LZ}$ and $\mathbf{LS}$.

It is easy to see $g\mathbf{N}$ ($\mathbf{N}$ being the pseudovariety of nilpotent semigroups) is defined by the pseudoidentity of Fig. 3. Certainly the pseudoidentity of Fig. 3 is satisfied by $g\mathbf{N}$. For the converse, one merely takes the canonical relational morphism of an X-generated semigroupoid C with the free nilpotent semigroup of index $|E(C)| + 1$, on generators $E(X)$, and checks it must be a division. If we force the loops to be idempotents in the above graph with no other relations, we can get a locally nilpotent semigroupoid which is not globally nilpotent. Then using a similar argument to the above one for $\mathbf{RS}$ and that $\mathbf{N} * \mathbf{G} = \mathbf{N} \vee \mathbf{G}$, we see that the pseudovariety $\mathbf{E1} = \mathbf{N} \vee \mathbf{G}$ is defined by the pseudoidentity of Fig. 4.

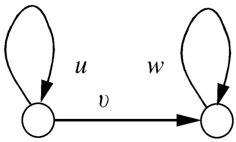
By forcing the loops in the graph of Fig. 4 to be idempotent but adding no other restrictions we get a semigroup locally in $\mathbf{N} \vee \mathbf{G}$, but not globally. This shows that in Proposition 10.2, some hypotheses are needed on $\mathbf{V}$.

We now obtain a pseudoidentity defining $g\mathbf{CS}$, where $\mathbf{CS}$ is the pseudovariety of completely simple semigroups, by using that $\mathbf{CS} = \mathbf{G} * \mathbf{RZ}$.



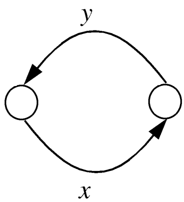
$$uv^{\omega}w = u'(v')^{\omega}w'$$

Fig. 3. Pseudoidentity for $g\mathbf{N}$.



$$u^{\omega}v = vw^{\omega}$$

Fig. 4. Pseudoidentity for $g(\mathbf{N} \vee \mathbf{G})$.



$$x(yx)^{\omega} = x$$

Fig. 5. Pseudoidentity for $g\mathbf{CS}$.

Proposition 10.9. *$g\mathbf{CS}$ is defined by the pseudoidentity of Fig.* 5

Proof. Clearly this pseudoidentity is satisfied by $g\mathbf{CS}$. For the converse, let C be a semigroupoid, generated by a finite graph X, satisfying the above pseudoidentity. We will use that $g\mathbf{CS} = g\mathbf{G} * g\mathbf{RZ}$. Let φ be the canonical relational morphism of C with the free right zero semigroup on $E(X)$. We show that the idempotents of D_{φ} are semigroupoid identities. It will then follow, since $\mathbf{G}$ is local [20] as a pseudovariety of monoids, that the derived semigroupoid is in $g\mathbf{G}$. We note, idempotents must be of the form $(((c_1,c_2),x),(fx,x))$ with $fx:c_2\to c_2$ idempotent. But the only arrows with endpoint $((c_1,c_2),x)$ are of the form $(((c_1,c_3),z),(f'x,x))$. But pseudoidentity of Fig. 5 implies $f'xfx=f'x$ and thus the idempotents of D_φ are right semigroupoid identities. To see that they are left identities, note that any element of $C(c_1,c_2)\cap x\varphi^{-1}$ is of the form $f'x$. Hence by the assumption that C satisfies the pseudoidentity of Fig. 5, we see $f'xfx=f'x$. Thus the identifications performed in constructing D_φ force the idempotent to be a left identity. $\square$

By taking the free semigroupoid on the above graph with the relations $(yx)^2=yx$ and $(xy)^2=xy$, we get a semigroupoid which is locally completely simple but not globally.

11. Applications to groupoids

We now present some applications to the theory of groupoids, and in particular, to the study of extensions. A *groupoid* is a category G with an additional operation $(\ )^{-1}:E(G)\to E(G)$ such that

- $f^{-1}\alpha=f\omega,\ f^{-1}\omega=f\alpha.$
- $ff^{-1}=1_{f\alpha},\ f^{-1}f=1_{f\omega}.$

Note $(f^{-1})^{-1}=f$ and any morphism of categories automatically preserves $(\ )^{-1}$. One defines a relational morphism of groupoids to be a subgroupoid of the product which projects onto the first factor as a quotient morphism. One can then define (pseudo) varieties of groupoids. But we will see momentarily, that these correspond precisely to (pseudo) varieties of groups. One can also take the semidirect product, as defined earlier, of two groupoids G and G' and one gets a groupoid. Indeed if $g':d_1\to d_2\in G''$, $g:d_1\psi\to g'd_2\psi\in G$, then $(g,g')^{-1}=((g')^{-1}(-g),(g')^{-1})$, where we write G additively.

Let G be a groupoid, $c\in V(G)$. We denote the local group at c by G_c. We say a groupoid is *connected* if the underlying graph is connected. Note any two vertices of a connected groupoid are connected by an edge. The following is then well known.

Proposition 11.1. *Let G be a connected groupoid, then for any $c,c'\in V(G)$, $G_c\cong G_{c'}$ via conjugation by any arrow $g:c\to c'$.*

The following, in the case of category divisions, is in [20].

Proposition 11.2. *Let G be a connected groupoid, $c\in V(G)$. Then $G_c<G$ and $G<G_c$.*

Proof. Clearly, $G_c<G$. For the converse, choose for each $d\in V(G)$ an arrow $g_d:c\to d$. Define a morphism $\varphi:G\to G_c$ by the obvious map on vertices and on arrows by $f:\ c_1\to c_2\mapsto g_{c_1}fg_{c_2}^{-1}$. This is clearly a faithful morphism of groupoids. $\square$

Since (pseudo) varieties of groupoids are closed under coproducts ($\coprod G_i < \prod G_i$). we thus see from the above proposition that (pseudo) varieties of groupoids correspond exactly to (pseudo) varieties of groups.

When studying quotient morphisms of groupoids, clearly it is only necessary to study the connected case. Hence from now on, we will assume all groupoids are connected. We now discuss extensions of groupoids and show that split extensions correspond exactly to semidirect products of groups with groupoids where the groupoid acts on the group. We feel this is further justification for our definition of the semidirect product of categories.

Let G be a groupoid, $c \in V(G)$, $K \lhd G_c$ a normal subgroup. We associate a congruence $\equiv_K$ to K as follows. For each $d \in V(G)$ choose an arrow $g_d : c \to d$. Then for $f, f' : c_1 \to c_2$ we define $f \equiv_K f'$ if $g_{c_1} f (f')^{-1} g_{c_1}^{-1} \in K$.

Proposition 11.3. *The above congruence is indeed a well-defined congruence independent of the choices of the g_d.*

Proof. We first show independence of the choices of the g_d. Suppose $g_{c_1}, g'_{c_1} : c \to c_1$. Then since K is normal, $g_{c_1}(g'_{c_1})^{-1} K g'_{c_1} g_{c_1}^{-1} = K$, so $(g'_{c_1})^{-1} K g'_{c_1} = g_{c_1}^{-1} K g_{c_1}$. But this says precisely that $g_{c_1} f (f')^{-1} g_{c_1}^{-1} \in K$ if and only if $g'_{c_1} f (f')^{-1} (g'_{c_1})^{-1} \in K$.

Clearly, $\equiv_K$ is an equivalence relation. We now show it is a right congruence, the proof that it is a left congruence is dual. Suppose $h : c_1 \to c_2, f, f' : c_2 \to c_3, f \equiv_k f'$. Then $g_{c_1} h : c \to c_2$. So by the independence result above,

$$g_{c_1} h f (h f')^{-1} g_{c_1}^{-1} = (g_{c_1} h) f (f')^{-1} (g_{c_1} h)^{-1} \in K. \qquad \square$$

We now show that all congruences of groupoids arise in this manner.

Proposition 11.4. *Let G be a groupoid, $\equiv$ a congruence, $\rho : G \to G/\equiv$ the quotient map. Let $c \in G$. Let K be the equivalence class of 1_c in G_c. Then K is a normal subgroup and $\equiv\ =\ \equiv_K$. Note that up to conjugation, and hence isomorphism, K is independent of c. We are thus justified in calling K the kernel of φ, $\ker \varphi$.*

Proof. Suppose $f, f' : c_1 \to c_2, f \equiv f'$. Then if $g_{c_1} : c \to c_1$, we see that $g_{c_1} f (f')^{-1} g_{c_1}^{-1} \equiv 1_c$. Conversely, if $g_{c_1} f (f')^{-1} g_{c_1}^{-1} \in K$, then $f (f')^{-1} \equiv g_{c_1}^{-1} g_{c_1} = 1_{c_1}$, so $f \equiv f'$. The rest is clear. $\square$

We note that φ is an isomorphism if and only if $\ker \varphi = 1$.

We say a groupoid G is an *extension* of a group K by G' if there exists a quotient map $\varphi : G \to G'$ such that $\ker \varphi = K$. We will write extensions, following the notation of group theory, by

$$1 \to K \to G \to G' \to 1.$$

We say the extension is *split* if there is a morphism $\rho : G' \to G$ such that $\rho\varphi = id_{G'}$. We call ρ the *splitting map*. For notational purposes, we will write all groups and groupoids multiplicatively from this point on. Left actions will then be written by exponentiation, that is the action say of g on k will be written ${}^{g}k$. Note, any partial full endofunctor of a one vertex category must be an endomorphism. In particular we see that a groupoid acts on a group via automorphisms.

Lemma 11.5. *Let K be a group, G a groupoid which acts on the left of K (by automorphisms). Then the projection gives rise to a split extension*

$$1 \to K \to K * G \to G \to 1$$

with splitting map given by $gp = (1, g)$.

Proof. We show ρ is a morphism, the rest is obvious. But if $g, g' \in E(G)$ are composable, $(1,g)(1,g') = (1, gg')$ since ${}^{g}1 = 1$, ${}^{g}(\ )$ being an automorphism.

Lemma 11.6. *Let $\varphi : G \to G'$ give rise to a split extension*

$$1 \to K \to G \to G' \to 1$$

with splitting map ρ. Then $G \cong K * G'$.

Proof. Without loss of generality; we can assume φ is the identity on objects and $K = G_c$ for $c \in V(G)$. Fix for each $d \in V(G)$ and edge $g_d : c \to d$. We will furthermore, choose $g_c = 1_c$. Define an action of G' on K by if $g' : c_1 \to c_2 \in G'$ and $k \in K$, then

$${}^{g'}k = g_{c_1}(g'\rho)g_{c_2}^{-1}kg_{c_2}(g'\rho)^{-1}g_{c_1}^{-1}.$$

It is easy to see that this is a left action. Each arrow of G' clearly gives an automorphism of K. Furthermore, if $g' : c_1 \to c_2$ and $g'' : c_2 \to c_3$ are arrows of G', then

$$\begin{aligned} {}^{g'}({}^{g''}k) &= g_{c_1}(g'\rho)g_{c_2}^{-1}g_{c_2}(g''\rho)g_{c_3}^{-1}kg_{c_3}(g''\rho)^{-1}g_{c_2}^{-1}g_{c_2}(g'\rho)^{-1}g_{c_1}^{-1} \\ &= g_{c_1}(g'\rho)(g''\rho)g_{c_3}^{-1}kg_{c_3}(g''\rho)^{-1}(g'\rho)^{-1}g_{c_1}^{-1} = {}^{g'g''}k. \end{aligned}$$

We define a morphism $\psi : K * G' \to G$ by the identity on objects and on arrows by $(k, g')\psi = g_{c_1}^{-1}kg_{c_1}(g'\rho)$, where $g' : c_1 \to c_2$. We claim ψ is an isomorphism. First we check it is a morphism. Suppose $g' : c_1 \to c_2$, $g'' : c_2 \to c_3 \in G'$ and $k, k' \in K$. Then

$$\begin{aligned} ((k,g')(k',g''))\psi &= (k\,{}^{g'}k', g', g'')\psi \\ &= (kg_{c_1}(g'\rho)g_{c_2}^{-1}k'g_{c_2}(g'\rho)^{-1}g_{c_1}^{-1}, g'g'')\psi \\ &= g_{c_1}^{-1}kg_{c_1}(g'\rho)g_{c_2}^{-1}k'g_{c_2}(g'\rho)^{-1}g_{c_1}^{-1}g_{c_1}(g'g'')\rho \\ &= g_{c_1}^{-1}kg_{c_1}(g'\rho)g_{c_2}^{-1}k'g_{c_2}(g''\rho) \\ &= (k,g')\psi(k',g'')\psi. \end{aligned}$$

To see the morphism is full, note if $g : c_1 \to c_2 \in G$, then $k = g_{c_1}g(g\varphi\rho)^{-1}g_{c_1}^{-1} \in K$ and $(k, g\varphi)$ maps to g. Since ψ is a quotient morphism, to show it is an isomorphism, we just need to show $\ker\psi = 1$. We compute the kernel at c. Suppose g': $c \to c$

and $(k,g')\psi = 1_c$. Then since $g_c = 1_c$, we have $k(g'\rho) = 1_c$ and so $(g'\rho)\varphi = 1_c$. But $\rho\varphi = id_{G'}$ so $g' = 1_c$ and hence $k = 1_c$. Thus ker $\psi = 1$. $\square$

As a consequence, we have the following generalization of a well known theorem from group theory.

Theorem 11.7. *There is a split extension of groupoids*

$$1 \to K \to G \to G' \to 1$$

if and only if $G \cong K * G'$.

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